Linear Analysis of Coupled Equations for Sediment Transport

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1. INTRODUCTION
A number of stability analyses have been performed to study the one-dimensional St. Venant and Exner equations over an erodible bed (e.g. de Vries, 1965; Gradowczyk, 1968; Lai, 1991; Sloff, 1993). Typically such analyses are used to find instabilities resulting in the spontaneous formation of bedforms. In an unmodified model, all disturbances associated with the bed celerity are stable. Here such a model is reanalyzed in light of the recent interesting results of Lisle et al. (1996), who found that a long hump introduced into a river may disperse in place rather than translate either downstream or upstream. The conditions under which dispersion dominates translation are analyzed here in terms of a fully coupled linear model.

2. GOVERNING EQUATIONS
The 1-D governing equations are the shallow water and Exner equations,

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial s} = -g \frac{\partial}{\partial s} \left( \eta + H \right) - \frac{\tau_b}{\rho H}, \quad \frac{\partial H}{\partial t} + \frac{\partial}{\partial s} (UH) = 0, \quad \frac{\partial \eta}{\partial t} + \frac{1}{1 - \lambda_p} \frac{\partial q}{\partial s} = 0 \quad \text{(1a,b,c)}
\]

where \( U \) = flow velocity, \( H \) = flow depth, \( \eta \) = bed elevation, \( \rho \) = water density, \( q \) = sediment transport rate per unit width, \( \tau_b \) = bed shear stress, \( \lambda_p \) = porosity \( s \) = the streamwise coordinate and \( t \) = time. These equations are closed with relations for resistance and bedload transport. Here the Keulegan resistance relation is used and bedload transport is taken to be a function of bed shear stress,

\[
\tau_b = \rho C_f U^2, \quad C_f^{-1/2} = 2.5 \ell / \left( 11 H / k_s \right), \quad q = q(\tau_b) \quad \text{(2a,b,c)}
\]

where \( k_s \) is roughness height. Suspended load is neglected for coarse-bedded streams.

3. BASE STATE SOLUTION AND NON-DIMENSIONALIZATION
For a given water discharge per unit width \( q_w \) and bed slope \( S \), an exact solution exists to the governing equations. A small perturbation is introduced into this base state solution below. The base state solution is

\[
U_0 H_0 = q_w, \quad q_0 = q(\tau_{b0}), \quad \eta_0 = \eta_i - S s \quad \text{(3a,b,c)}
\]

\[
\tau_{b0} = \rho C_{f0} U_0^2 = \rho g H_0 S, \quad C_{f0}^{-1/2} = 2.5 \ell / \left( 11 H / k_s \right) \quad \text{(3d,e)}
\]
where \( \eta_i \) = a given bed elevation at \( s = 0 \).

The variables are made non-dimensional in the following fashion:

\[
\begin{align*}
\tilde{s} &= \frac{s}{H_0}, \quad \tilde{t} = \beta U_0 t / H_0, \quad \tilde{U} = U / U_0, \quad \tilde{H} = H / H_0 \\
\tilde{\eta} &= (\eta - \eta_0) / H_0, \quad \tilde{\tau}_b = \tau_b / \tau_{b0}; \quad \tilde{q} = q / q_0
\end{align*}
\]

(4a,b,c,d)

Substituting (4a,b,c,d,e,f,g) into the governing equations, the following dimensionless forms are obtained,

\[
\begin{align*}
\beta \frac{\partial \tilde{U}}{\partial \tilde{t}} + \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{s}} &= C_{f0} - F^{-2} \frac{\partial}{\partial \tilde{s}} (\eta + \tilde{H}) - C_{f0} \frac{\tilde{\tau}_b}{H} \\
\beta \frac{\partial \tilde{H}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{s}} (\tilde{U} \tilde{H}) &= 0, \quad \frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \frac{\partial \tilde{q}}{\partial \tilde{x}} = 0
\end{align*}
\]

(5a,b,c)

where \( F \) is the base state Froude number and \( \beta \) is the base state ratio of sediment transport rate to water discharge (corrected for porosity);

\[
F = \frac{U_0}{\sqrt{g H_0}}, \quad \beta = \frac{q_0}{(1 - \lambda_p) U_0 H_0}
\]

(6a,b)

The base state solution thus corresponds to

\[
\left( \tilde{U}, \tilde{H}, \tilde{\eta}, \tilde{\tau}_b, \tilde{q}, \tilde{\eta} \right) = (1, 1, 1, 1, 0)
\]

(7)

4. LINEARIZATION FOR SMALL-AMPLITUDE PERTURBATIONS

The following perturbations are introduced to the system:

\[
\left( \tilde{U}, \tilde{H}, \tilde{\tau}_b, \tilde{q}, \tilde{\eta} \right) = (1 + u', 1 + h', 1 + \tau', 1 + q', 0 + \eta')
\]

(8)

Substituting (8) into (5a,b,c), dropping \( \sim \) on \( t \) and \( s \) for simplicity and retaining only linear terms, it is found that

\[
\begin{align*}
\beta \frac{\partial u'}{\partial t} + \frac{\partial u'}{\partial s} &= -F^{-2} \frac{\partial}{\partial s} (u' + h') - C_{f0} \left[ 2u' - (1 + M)h' \right] \\
\beta \frac{\partial h'}{\partial t} + \frac{\partial}{\partial s} (u' + h') &= 0, \quad \frac{\partial \eta'}{\partial t} + N \frac{\partial}{\partial s} (2u' - Mh') = 0
\end{align*}
\]

(9a,b,c)

where the order-one dimensionless coefficients \( M \) and \( N \) are given by

\[
M = 5C_{f0}^{1/2}, \quad N = \frac{\tau_{b0}}{q_0} \frac{dq}{d\tau_b}
\]

(10a,b)

Now assume that

\[
\eta' = \eta_1 e^{ik(s-ct)}, \quad u' = u_1 e^{ik(s-ct+\theta_u)}, \quad h' = h_1 e^{ik(s-ct+\theta_h)}
\]

(11a,b,c)

where

\[
k = \frac{2\pi H_0}{\lambda}
\]

(12)

is a dimensionless wave number and \( \lambda \) is the wavelength of the perturbation. Here \( \eta_1, u_1, h_1 \ll 1 \) are real numbers denoting the amplitudes of the perturbations in channel bed, flow velocity and flow depth, and \( \theta_u \) and \( \theta_h \) denote phase differences
of perturbations of the flow velocity and depth relative to the bed perturbation. Also, \( c \) is a complex celerity the real and imaginary parts of which denote wave speed and growth rate of the perturbation, both non-dimensionalized with \( \beta U_0 \). Here a negative growth can be interpreted as a tendency for the bedform to disperse, and is thus called the dispersion coefficient here. Substituting (11a,b,c) into (9a,b,c), it is found that

\[
\left( \beta c - 1 \right) u_* e^{i \theta_0} - F^{-2} \left( \eta_* + h_* e^{i \theta_0} \right) + i \frac{C_{f_0}}{k} \left[ 2 u_* e^{i \theta_0} - (1 + M) h_* e^{i \theta_0} \right] = 0 \quad (13a)
\]

\[
\beta ch_* e^{i \theta_0} - \left( u_* e^{i \theta_0} + h_* e^{i \theta_0} \right) = 0, \quad c \eta_* - N \left( 2 u_* e^{i \theta_0} - M h_* e^{i \theta_0} \right) = 0 \quad (13b,c)
\]

\[
\frac{u_* e^{i \theta_0}}{\eta_*} = \frac{(1 - \beta c)c}{N \left[ M + 2(1 - \beta c) \right]}, \quad \frac{h_* e^{i \theta_0}}{\eta_*} = -\frac{c}{N \left[ M + 2(1 - \beta c) \right]} \quad (14a,b)
\]

\[
\left[ \beta c - \frac{2}{3} \left(1 - i \frac{C_{f_0}}{k} \right) \right]^3 + \Phi \left[ \beta c - \frac{2}{3} \left(1 - i \frac{C_{f_0}}{k} \right) \right] + \Psi = 0 \quad (14c)
\]

where

\[
\Phi = -\frac{1}{3} \left[ 1 + 3F^{-2} (1 + 2\beta N) - 4 \left( \frac{C_{f_0}}{k} \right)^2 + i(1 + 3M) \frac{C_{f_0}}{k} \right] \quad (15a)
\]

\[
\Psi = \frac{2}{27} - \frac{1}{3} F^{-2} \left[ 2 - \beta N(2 + 3M) \right] - \frac{2}{9} (1 + 3M) \left( \frac{C_{f_0}}{k} \right)^2 - \frac{1}{9} \left[ 4 + 3M - 3F^{-2} (1 + 2\beta N) + \frac{8}{3} \left( \frac{C_{f_0}}{k} \right)^2 \right] \frac{C_{f_0}}{k} \quad (15b)
\]

The solutions of (14c) are

\[
\beta c_j = \frac{2}{3} \left(1 - i \frac{C_{f_0}}{k} \right) + e^{\frac{2(j-1)\pi}{3}} \Omega_+ + e^{-\frac{2(j-1)\pi}{3}} \Omega_-, \quad j = 1, 2, 3 \quad (16a,b,c)
\]

where

\[
\Omega_+ = \left\{ \frac{\Psi}{2} \pm \sqrt{\left( \frac{\Psi}{2} \right)^2 + \left( \frac{\Phi}{3} \right)^3} \right\} \quad (17a,b)
\]

In particular, \( c_2 = c_* + ic_i \) is a complex eigenvalue, where \( c_* \) is dimensionless bed wave speed and \( c_i \) is dimensionless dispersion coefficient of the bed perturbation. Also \( c_1 \) and \( c_3 \) are complex eigenvalues associated with the flow. Here \( c_* \) and \( c_i \) are evaluated numerically. For sufficiently small values of the ratio \( C_{f_0} / k \) \((\leq 0.65)\), (16a,b,c) give accurate results. However as \( C_{f_0} / k \) becomes larger the difference between the three roots becomes larger and (16a,b,c) is no longer accurate. In this case, a Newton-Raphson method can be applied to solve (14c).

5. DISCUSSION OF THE RESULTS
If the friction terms in the momentum equation are neglected, (14) reduces to the form given in Cui et al. (1995). In this case, there are only real solutions and they represent wave speeds of the perturbation. In treating the more general case, most analyses employ the quasi-steady assumption according to which the equations of
flow momentum and mass balance are decoupled from the Exner equation by neglecting the $\frac{\partial}{\partial t}$ terms in the former two. The validity of this assumption can be seen from (9a,b,c) to be justified as $\beta << 1$. In this case it is found that

$$
c_r = \frac{N(2 + M)(1 - F^2)}{(1 - F^2)^2 + [(3 + M)F^2C_{fo} / k]^2}, \quad c_i = \frac{-N(2 + M)(3 + M)F^2C_{fo} / k}{(1 - F^2)^2 + [(3 + M)F^2C_{fo} / k]^2} \tag{18a,b}
$$

In the present analysis, however, the quasi-steady assumption is not used except as a special case.

The effect of sediment transport rate, or the value of $\beta$, on the bed wave speed $c_r$ and dispersion coefficient $c_i$ is shown in Fig. 1, in which $N = 3.0$, $k = 0.1$ and $C_{fo} = 0.005$. Note that $c_i$ is always negative, indicating that bedforms disperse rather than grow; $c_r$ is positive or negative depending on whether the flow is subcritical or supercritical. Note, however that the ratio $|c_i/c_r|$ is less than unity except in a small band near $F = 1$. The zone where $|c_i/c_r| > 1$ vanishes when $\beta$ becomes large enough to cause the quasi-steady assumption to fail. The implication is that dispersion dominates wave translation only in a small range corresponding to near-critical flow and low sediment transport rates.

In Fig. 2, however, wavenumber $k$ is allowed to vary while $N$, $C_{fo}$ and $\beta$ are held constant at 3.0, 0.005 and $1 \times 10^{-3}$, respectively. It is seen that as $k$ becomes small, the range over which $|c_i/c_r| > 1$ widens to the point where dispersion can be expected to be dominant for all flows except highly subcritical ones.

The above analysis thus indicates the conditions under which a 1-D bedform might predominantly disperse in place rather than translate as a wave. The sediment transport rate should be low in terms of small $\beta$, the Froude number $F$ should not be too far from unity and the wavelength of the bedform should be very large compared to depth, so that $k << 1$. These conditions are realized in the experiment reported by Lisle et al (1996).

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REFERENCE


Fig. 1  The effect of $\beta$ on bed celerity and dispersion coefficient
Fig. 2 The effect of wave number $k$ on bed celerity and dispersion coefficient